MATH 7210/6710 — HOMEWORK 2 — SEPTEMBER 18, 2020

Exercise 1. Let f be a measurable function defined on $\mathbb{R}^d$. Show that there exists a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ such that $|\phi_k(\mathbf{x})| \le |\phi_{k+1}(\mathbf{x})|$, $|\phi_k(\mathbf{x})| \le |f(\mathbf{x})|$, and $\lim_{k\to\infty} \phi_k(\mathbf{x}) = f(\mathbf{x})$, for all $k \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^d$.

Exercise 2. Show by counterexample that the condition $m(E) < \infty$ in Egorov's theorem can not be removed.

Exercise 3. Suppose that f and $\{f_k\}_{k=1}^{\infty}$ are measurable functions defined on $[a, b]$ and

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{a.e. } x \in [a, b].$$

Show that there exist $E_n \subset [a, b]$ $(n = 1, 2, \cdots)$ such that

$$m\Big([a, b] \setminus \cup_{n=1}^{\infty} E_n\Big) = 0,$$

and $\{f_k(x)\}$ converges to $f(x)$ uniformly on each E_n.

(**Hint**: Egorov's theorem)

Exercise 4. Let f be a measurable function defined on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$. Suppose that f is finite-valued almost everywhere on E. Show that for any $\varepsilon > 0$, there exists a bounded and measurable function g on E such that

$$m(\{x \in E : |f(x) - g(x)| > 0\}) < \varepsilon.$$

(**Hint**: Define $E_n = \{x \in E : |f(x)| \ge n\}$, $n = 1, 2, \cdots$, and $E_\infty = \{x \in E : |f(x)| = \infty\}$)

Exercise 5. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be measurable functions defined on $E \subset \mathbb{R}^d$. Suppose that $\{f_n\}$ and $\{g_n\}$ converge to 0 in measure on E. Show that $\{f_n \cdot g_n\}$ converges to 0 in measure on E.

Due: October 2, 2020